ON THE DERIVATION OF THE EQUATION OF MOTION IN A SCALAR MODEL.

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General Relativity (GR) is unique among the class of field theories in the treatment of the equations of motion. The equations of motion of massive particles are completely determined by the field equation.
It turns out that most field equations are linear in the second order derivatives and quadratic in the first order derivatives both with coefficients that depend only on the field variables. For the purpose of exposition let us write the general field equation, symbolically, as

$$aA(\Phi) + bB(\nabla\Phi, \nabla\Phi) = 0, \tag{1}$$

where A is a second order linear operator, while B is a quadratic form. The coefficients a and b are the dimensionless functions of the field variable (or constants) $a = a(\Phi),\ b = b(\Phi)$.
The covariance condition as well as the existence of an action functional provide severe restrictions on the coefficient $a(\Phi)$ and $b(\Phi)$. Note that Einstein equation does have the form (1).
A novel algorithm for the derivation of equations of motion is worked. It is for field equations that are Lorentz invariant. A condense summary of the algorithm follows.

1 Compute a static, spherically symmetric solution F of the field equation. It will be singular at the origin. This will be taken to be the field generated by a single particle.

2 Move the solution on a trajectory $\boldsymbol{\psi}(t)$ by apply the instantaneous Lorentz transformation based on $\dot{\boldsymbol{\psi}}(t)$.

3 Take the field generated by n particles to be the superposition of the fields generated by the single particles.

4 Compute the leading (linear) part of the equation. Hopefully, only terms that involves $\ddot{\boldsymbol{\psi}}$ will be dominant. It turns out that these terms are *the agent of inertia*.

5 Compute the "force" between the particles by the quadratic part of the equation. By **3** it will be

$$bB(\nabla \sum {}^{(j)}F, \nabla \sum {}^{(k)}F).$$

Since

$$aA({}^{(j)}F) + bB(\nabla^{(j)}F, \nabla^{(j)}F) = 0$$

it follows that

$$bB(\nabla^{(j)} F, \nabla^{(k)} F) \qquad \text{with} \qquad j \neq k$$

remains.

6 Equate for each singularity, the highest order terms of the singularities that came from the linear part and the quadratic parts, respectively. This is an equation between the inertial part and the force. For the j-th singularity the linear terms will contribute $\ddot{\boldsymbol{\psi}}_j$, while the quadratic part will include all the terms with fixed j. Equating of the two highest order singularities should, hopefully, result in Newton's law of attraction.

The proposed method was applied to the vacuum Einstein field equation.

$$R_{ik} = 0.$$

In the first order approximation the metric is determined by a unique scalar function f. The field equation is transformed to a nonlinear scalar field equation

$$d * df = kdf \wedge *df, \tag{2}$$

or, in coordinate form

$$\Box f = k\eta^{ab} f_{,a} f_b. \tag{3}$$

The algorithm above resulted, indeed, in the verification of Newton gravitation law. The trajectories obtained above are approximate. At this point, two avenues are open. The first one, which is adopted by EIH is to get higher order approximations to the trajectories. This procedure is also used in the PPN approach. By these methods, the successive approximations become highly singular near the particle trajectories.
The second avenue is to embed the singularities in a field satisfying the field equations. For that purpose, the successive approximations should add, near the trajectories, regular terms and, possibly, low order singular terms as well. In the first order approximation the trajectories are taken to be fixed and the calculation is done only for the first order terms. Only the scalar model will be considered.
The desired field will be a solution φ of (3),

$$\varphi = e^{-kf+g}, \tag{4}$$

where f is defined by the singular solution and g is regular at all the singularities ("multi Green function"). Thus

$$\Box g - 2k\eta^{ab} f_{,a} g_{,b} + \eta^{ab} g_{,a} g_{,b} = -F, \tag{5}$$

where

$$k(\Box f - k\eta^{ab} f_{,a} f_{,b}) = F \tag{6}$$

Let us do some formal reasoning. The highest order singular terms are, near the j-th trajectory, $\mathcal{O}(|x - \psi(t)|^{-2})$. Therefore $F = \mathcal{O}(|x - \psi(t)|^{-1})$, the derivatives $f_{,a}$ are $\mathcal{O}(|x - \psi(t)|^{-2})$ Thus $g_{,a}$ should be $\mathcal{O}(|x - \psi(t)|)$ - a regular term.
Since f is singular the construction is not straightforward.